Triangular Distributions

A triangular distribution is a continuous probability distribution with a probability density function shaped like a triangle. It is defined by three values: the minimum value *a*, the maximum value *b*, and the peak value *c*.

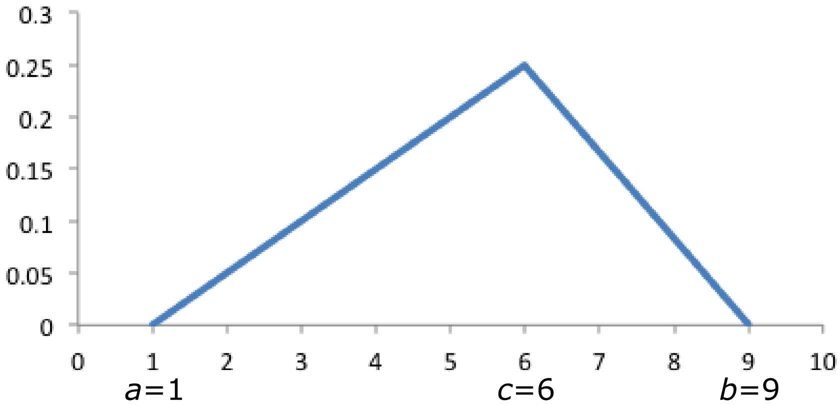
This is really handy as in a real-life situation we can often estimate the maximum and minimum values, and the most likely outcome, even if we don't know the mean and standard deviation.

The triangular distribution has a definite upper and lower limit, so we avoid unwanted extreme values. In addition the triangular distribution is a good model for skewed distributions. The sum of two dice is often modelled as a discrete triangular distribution with a minimum of 2, a maximum of 12 and a peak at 7.

Probability Density Function

All probability density functions have the property that the area under the function is 1. For the triangular distribution this property implies that the maximum value of the probability distribution function is $\frac{2}{(b-a)}$. It occurs at the peak value of *c*.

The probability density function of a triangular distribution is zero for values below *a* and values above *b*. It is piecewise linear rising from 0 at *a* to $\frac{2}{(b-a)}$ at *c*, then dropping down to 0 at *b*. The graph below shows the probability density function of a triangle distribution with *a*=1, *b*=9 and *c*=6. The peak is at *c*=6 with a function value of 0.25.



The probability density function of a triangular distribution

The formula for the probability density function is

$$f(x) = \begin{cases} 0, & x < a \\ \dfrac{2(x-a)}{(b-a)(c-a)}, & a \le x \le c \\ \dfrac{2(b-x)}{(b-a)(b-c)}, & c \le x \le b \\ 0, & x > b \end{cases}$$

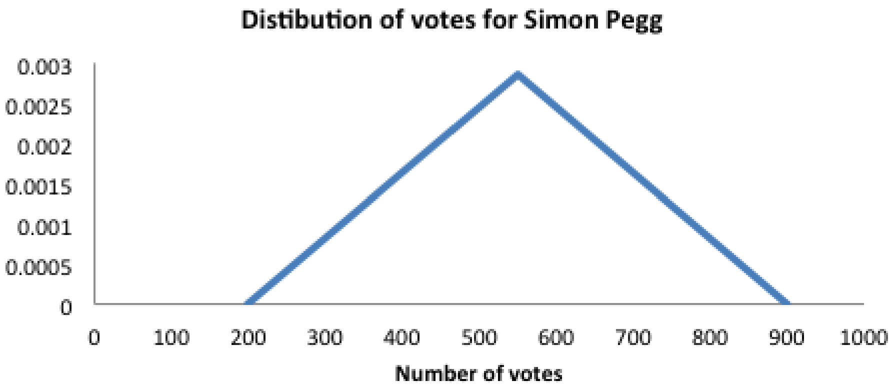
Example 1

A burger franchise planning a new outlet in Auckland uses a triangular distribution to model the future weekly sales. They estimate that the minimum weekly sales is $1000 and the maximum is $6000. They also estimate that the most likely outcome is around $3000. So they use *a*=$1000, *b*=$6000 and *c*=$3000. The graph of the probability density function reaches its maximum of 0.0004 at *c*=$3000. The graph of this probability density function is shown below.

The probability density function of weekly sales for a burger franchise

Example 2

Voting for the student representative on a school's Board of Trustees has closed but the votes have not been counted. Simon Pegg (a candidate) thinks about how many votes he thinks he will get. He thinks the most likely value is around 550, but he could get as many as 900 or as few as 200. Simon models the number of votes he may have received as a triangular distribution with minimum value *a*=200, maximum value *b*=900 and peak value *c*=550. The graph of the probability density function reaches its maximum of 0.002857 at *c*=550 and is shown below.



The probability density function of possible votes for Simon Pegg

Calculating probabilities

The probability density function is used to determine the probability that the random variable falls in some range. We want to determine the probability that the random variable is above a given value, below a given value, or between a pair of values. It is simply a matter of finding the area under the curve for the required interval.

For a triangular distribution this involves finding the area of one or two triangles and, possibly, a simple calculation. The process is as follows.

1. Determine which area is needed,
2. Determine which triangle(s) to use,
3. Calculate the area(s) of the triangle(s),
4. Calculate the probability sought.

Example 1

A burger franchise planning a new outlet in Auckland wants to determine the probability the new outlet will have weekly sales of less than $2000. If the weekly sales are less than this the outlet is unlikely to cover its costs.

So, they wish to calculate $\Pr(X < 2000)$. They use a triangular distribution to model the future weekly sales with a minimum value of *a*=$1000, and maximum value of *b*=$6000 and a peak value of *c*=$3000.

Step 1.
The area needed is shown as the shaded area in the graph below. That is, $\Pr(X < 2000)$ is the area under the probability density function for all values of *X* below 2000. In this case the probability sought is the area of a triangle.

Step 2.
The shaded area is the triangle to use.

Area corresponding to $\Pr(X < 2000)$

Step 3.
The area of a triangle is $area = \frac{1}{2} base \times height$. In this case $base = 2000 - 1000 = 1000$ and

the height is given by $f(2000)$ using the probability density function formula. Since 2000 is between *a*=1000 and *c*=3000 we have:

$$height = f(2000) = \frac{2(2000-a)}{(b-a)(c-a)} = \frac{2\times(2000-1000)}{(6000-1000)\times(3000-1000)} = 0.0002$$

So $area = \frac{1}{2}\times 1000\times 0.0002 = 0.1$

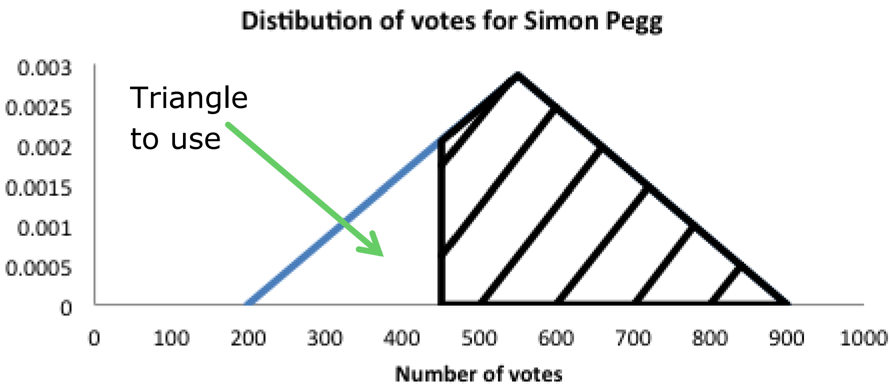
Step 4.
The probability is just the area of the triangle, so $\Pr(X < 2000) = 0.1$ (or 10%).

Example 2

Voting for the student representative on a school's Board of Trustees has closed but the votes have not been counted. Candidate Simon Pegg wants to determine the probability that he received more than 450 votes. This means Simon wants to determine $\Pr(X > 450)$. He models the number of votes he may have received as a triangular distribution with minimum value *a*=200, maximum value *b*=900 and peak value *c*=550.

Step 1.
$\Pr(X > 450)$ is the area under the probability density function for all values of *X* above 450. This area is shown as the shaded area in the graph below. In this case the shaded area is not a triangle.

Step 2.
The triangle to use is the **unshaded** area under the curve. Since the total area under the curve is 1, we can calculate $\Pr(X > 450)$ as $\Pr(X > 450) = 1 - area$.



Area corresponding to $\Pr(X > 450)$

Step 3.
The area of a triangle is $area = \frac{1}{2}base \times height$. For the triangle $base = 450 - 200 = 250$ and the height is $f(450)$. Since 450 is between *a*=200 and *c*=550 we have:

$$height = f(450) = \frac{2\times(450-200)}{(900-200)\times(550-200)} = 0.0020408$$

So $area = \frac{1}{2}\times 250\times 0.0020408 = 0.2551$

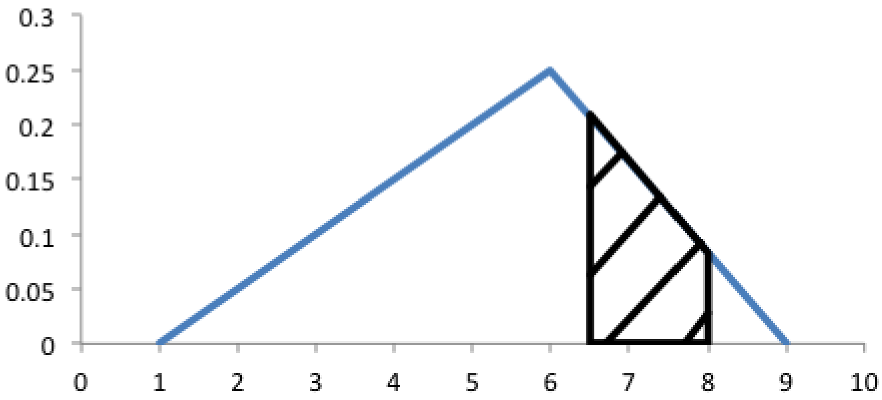
Step 4.
The probability is $\Pr(X > 450) = 1 - area = 1 - 0.2551 = 0.7449$ (or 74.5%).

Example 3

Find $\Pr(6.5 < X < 8)$ for a triangular distribution with minimum value *a*=1, maximum value *b*=9 and peak value *c*=6.

Step 1.
$\Pr(6.5 < X < 8)$ is the area under the probability density function for values of X between 6.5 and 8. This area is shown as the shaded area in the graph below. In this case the shaded area is not a triangle.

Step 2.
In this case, two triangles are needed. The smaller triangle is immediately to the right of the shaded area, with base between 8 and 9. The larger triangle includes the shaded area and the smaller triangle. It has a base between 6.5 and 9. The shaded area, $\Pr(6.5 < X < 8)$, is the area of the larger triangle minus the area of the smaller triangle.



Area corresponding to $\Pr(6.5 < X < 8)$

Step 3. For the smaller triangle $base_1 = 9 - 8 = 1$ and

$$height_1 = f(8) = \frac{2(b-8)}{(b-a)(b-c)} = \frac{2 \times 1}{8 \times 3} = 0.08333$$

So $area_1 = \frac{1}{2} base_1 \times height_1 = \frac{1}{2} \times 1 \times 0.08333 = 0.04167$. Then, for the larger triangle $base_2 = 9 - 6.5 = 2.5$ and

$$height_2 = f(6.5) = \frac{2 \times (9 - 6.5)}{(9-1) \times (9-6)} = 0.20833$$

So $area_2 = \frac{1}{2} \times 2.5 \times 0.20833 = 0.26042$.

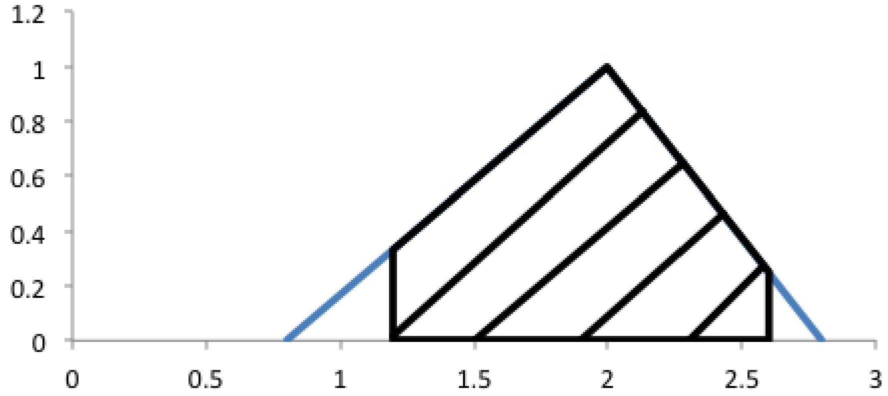
Step 4.
The probability is $\Pr(6.5 < X < 8) = area_2 - area_1 = 0.17708$ (or 17.7%).

Example 4

Find $\Pr(1.2 < X < 2.6)$ for a triangular distribution with minimum value *a*=0.8, maximum value *b*=2.8 and peak value *c*=2.0.

Step 1.
$\Pr(1.2 < X < 2.6)$ is the area under the probability density function for values of *X* between 1.2 and 2.6. This area is shown as the shaded area in the graph below.

Step 2.
In this case, two triangles are needed. Triangle 1 is immediately to the left of the shaded area, with base between 0.8 and 1.2. Triangle 2 is immediately to the right of the shaded area, with base between 2.6 and 2.8. The shaded area, $\Pr(1.2 < X < 2.6)$, is the total area under the curve, which is 1, minus the areas of the two triangles.



Area corresponding to $\Pr(1.2 < X < 2.6)$

Step 3.
For triangle 1, $base_1 = 1.2 - 0.8 = 0.4$ and $height_1 = f(1.2) = 0.3333$. This gives $area_1 = \frac{1}{2} \times 0.4 \times 0.3333 \ = 0.06667$. For the triangle 2 $base_2 = 2.8 - 2.6 = 0.2$ and $height_2 = f(2.6) = 0.25,$ so $area_2 = \frac{1}{2} \times 0.2 \times 0.25 \ = 0.025$.

Step 4.
The probability is $\Pr(1.2 < X < 2.6) = 1 - area_1 - area_2 = 0.9083$ (or 90.8%).

Further activities and resources are provided on http://statslc.com/

Feedback

Every effort was made to ensure these notes are useful and correct. Suggestions for improvement and corrections are welcomed by the authors at n.petty@statsLC.com.

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